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LETTER TO THE EDITOR

Phase diagram of the non-Hermitian asymmetric XXZ spin chain

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Abstract. The low-lying excitations of the asymmetric XXZ spin chain are derived explicitly in the antiferromagnetic regime through the Bethe ansatz. It is found that a massless and conformal invariant phase with central charge $c = 1$ is separated from a massive phase by a line on which the low-lying excitations surprisingly scale with the lattice length as $\Delta E \sim N^{-1/2}$. The mass gap vanishes with an exponent $\frac{1}{2}$ as one approaches the massless phase. The connection with the asymmetric six-vertex model and some physical consequences are discussed.

It is believed, although not rigorously proven, that the phase diagram of a classical statistical model in d dimensions can be mapped onto that of a quantum Hamiltonian in $(d-1)$ dimensions [1]. In particular, singularities of the classical free energy should correspond to singularities of the quantum ground-state energy, while the mass gap of the latter should scale, near criticality, as the inverse correlation length in the dth direction of the classical model $m \sim \xi_d^{-1}$. Well known examples are the 2d Ising and symmetric six-vertex models with their associated Ising and XXZ spin chains. In most cases, tackling the lower dimension quantum version turns out to be simpler, and one therefore expects the asymmetric XXZ chain to be more tractable than the associated asymmetric six-vertex model (i.e. with external fields). The six-vertex model was originally introduced to describe ferroelectric and antiferroelectric phase transitions in hydrogen-bonded crystals [2]. Recently, there has been renewed interest in the solution of its asymmetric version [3, 4], due to its relation to models of crystals which describe the equilibrium shape of the interface between the coexisting solid and vapour phases [5, 6].

The model is defined as follows: on a 2d square lattice, place on each vertical (horizontal) edge an up (right) or down (left) arrow. The ice condition restricts the number of allowed configurations to six [7], each being assigned a Boltzmann weight $R_{\alpha\alpha'}^{\beta\beta'}(u)$ (see figure 1), so that the Yang–Baxter equations are satisfied and the transfer matrix

$$T(u)_{\{\underline{\alpha}\},\{\underline{\alpha'}\}} = \sum_{\{\underline{\beta}\}} \sum_{k=1}^{N} R_{\alpha_k \alpha'_k}^{\beta_k \beta_{k+1}}(u) \tag{1}$$

forms a commuting family, $[T(u), T(u')] = 0$ for any two different values of the spectral parameter u [8]. Introducing the Pauli matrices σ^z, $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm \mathrm{i}\sigma^y)$ and the vertical polarization operator $S^z = \sum_{k=1}^{N} \sigma^z$, the associated spin chain Hamiltonian is obtained from the so-called extremely anisotropic limit ($u \to 0$) of the transfer matrix

$$T(u) = \mathrm{e}^{VS^z}\overline{T}(u) \qquad H = -\log(\mathrm{e}^{VS^z}) - \sinh\gamma \frac{\mathrm{d}}{\mathrm{d}u} \log \overline{T}(u)|_{u=0} \tag{2}$$

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which gives

$$H = -\sum_{j=1}^{N}\left[\frac{\epsilon\cosh\gamma}{2}(1+\sigma_j^z\sigma_{j+1}^z) + \mathrm{e}^{\Psi}\sigma_j^+\sigma_{j+1}^- + \mathrm{e}^{-\Psi}\sigma_j^-\sigma_{j+1}^+\right] - V\sum_{j=1}^{N}\sigma_j^z \tag{3}$$

where $\epsilon = \pm 1$. Since $[H, T(u)] = 0$, both can be diagonalized by the same Bethe ansatz. Note that for real Ψ ($\neq 0$), (3) is non-Hermitian [9].

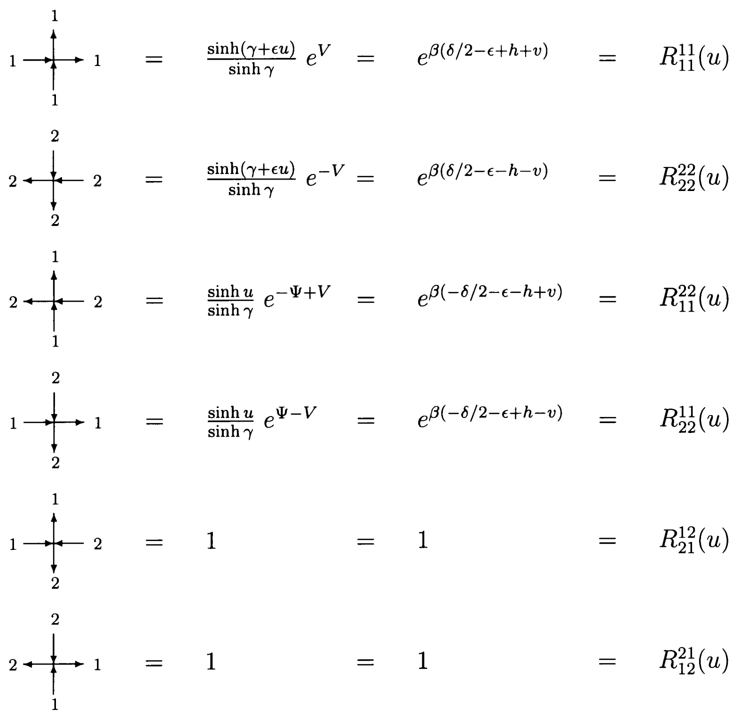

Figure 1. Boltzmann weights in the notation with spectral parameter u compared to that of [3]. Note that here $V = \beta(h+v)$ in contrast to the V of [3].

It is our aim in this letter to study the low-lying excitations of this asymmetric XXZ spin chain. Our motivation is twofold. On the one hand, the Bethe ansatz solution for the asymmetric six-vertex model was originally published in a very concise letter by Sutherland *et al* [10], and the phase diagram was determined from the analytical properties of the free energy. More details were published in [3, 11–13], but the critical behaviour has not been fully explored. On the other hand, the spin chain itself is of particular interest. In the ferromagnetic regime $\epsilon = 1$ and with the tuning $\Psi = \gamma$ (stochastic line), this Hamiltonian has been proposed as the time-evolution operator for a $1d$ two-species diffusion process $A + 0 \rightleftharpoons 0 + A$ [14, 15]. Under this condition H is also of significance in the understanding of Kardar–Parisi–Zhang-type growth phenomena [16]. Moreover, through a similarity transformation, it can be shown that (3) is equivalent to an XXZ model with boundary condition $\sigma_{N+1}^{\pm} = \exp(\pm\mathrm{i}\omega N)\sigma_1^{\pm}$ with $\omega = -\mathrm{i}\Psi$. The case of a real phase ω has been studied before [17]. For imaginary ω however, the physics is completely different.

In this letter we restrict ourselves to the antiferromagnetic regime $\epsilon = -1$, $V = 0$ but Ψ arbitrary. Note that $V = 0$ implies taking $h = -v$ in the six-vertex weights (figure 1). We summarize our results as follows: we have solved the Bethe ansatz in the thermodynamic limit and have found exact, analytic expressions for the ground-state energy, the mass gap and the dispersion relation. Our solution shows that a massive phase extends from $\Psi = 0$ to a critical $\Psi_c(\gamma)$ where the mass gap vanishes as $m \sim (\Psi_c - \Psi)^{\frac{1}{2}}$. On the transition line the spin chain is gapless with a dispersion relation at small momenta $\Delta E \sim |\Delta P|^{\frac{1}{2}}$, which indicates that the low-lying states scale, on a finite but large lattice, as $\Delta E \sim N^{-\frac{1}{2}}$. For $\Psi > \Psi_c$ we find a massless and conformal invariant phase, excitations scaling as $\Delta E \sim N^{-1}$ and a central charge $c = 1$ (Gaussian model). All these results have been checked through extensive numerical analysis.

Before studying the low-lying excitations, we present some known results about the Bethe ansatz equations. In a sector where n spins are flipped with respect to the reference state $|\uparrow\uparrow \ldots \uparrow\rangle$ [18], the Bethe ansatz [19] gives for the eigenvalues of (3) and their momentum

$$E = N\cosh\gamma - 2\sinh\gamma\sum_{k=1}^{n}\phi(\gamma;\alpha_k) - V(N-2n) \tag{4}$$

$$\phi(\gamma;\alpha) = \frac{\sinh\gamma}{\cosh\gamma - \cos\alpha}$$

$$\mathrm{e}^{\mathrm{i}P} = \mathrm{e}^{\Psi n}\prod_{j=1}^{n}\frac{\sinh((\gamma/2) - (\mathrm{i}\alpha_j/2))}{\sinh((\gamma/2) + (\mathrm{i}\alpha_j/2))} \tag{5}$$

where the rapidities $\{\alpha_k\}$ are solutions of the Bethe ansatz equations

$$\left[\frac{\sinh((\gamma/2) + (\mathrm{i}\alpha_k/2))}{\sinh((\gamma/2) - (\mathrm{i}\alpha_k/2))}\right]^N = (-1)^{n-1}\mathrm{e}^{\Psi N}\prod_{l=1}^{n}\frac{\sinh(\gamma + (\mathrm{i}/2)(\alpha_k - \alpha_l))}{\sinh(\gamma - (\mathrm{i}/2)(\alpha_k - \alpha_l))}. \tag{6}$$

Equation (4) is found by taking the logarithmic derivative, as in (2), of eigenvalues of the transfer matrix. In the following we take $V = 0$. After taking the logarithm of (6) we get

$$p^0(\alpha_l) - \frac{1}{N}\sum_{j=1}^{n}\Theta(\alpha_l - \alpha_j) = -\mathrm{i}\Psi + \frac{2\pi}{N}I_l \tag{7}$$

with

$$p^0(\alpha) = -\mathrm{i}\ln\left(\frac{\sinh((\gamma/2) + (\mathrm{i}\alpha/2))}{\sinh((\gamma/2) - (\mathrm{i}\alpha/2))}\right) \qquad \Theta(\alpha) = -\mathrm{i}\ln\left(\frac{\sinh(\gamma + (\mathrm{i}\alpha/2))}{\sinh(\gamma - (\mathrm{i}\alpha/2))}\right). \tag{8}$$

$\{I_l\}$ is a set of n integers (half-odd numbers) if n is odd (even). Summing (7) over l and taking the logarithm of (5), one gets for the momentum

$$P = -\sum_{l=1}^{n}\frac{2\pi I_l}{N}. \tag{9}$$

The branch cuts of $\Theta(\alpha)$ in the α-plane are taken from $2\mathrm{i}\gamma$ to $\mathrm{i}\infty$ and from $-2\mathrm{i}\gamma$ to $-\mathrm{i}\infty$ and we choose $\Theta(0) = 0$. The distribution of the integers I_l already determines a given state of H. It is known that at $\Psi = 0$ the antiferromagnetic ground state is defined by a closely packed sequence of $n_0 = N/2$ integers I_l^0 which are symmetrically distributed with repsect to zero, and a set of real rapidities distributed in $[-\pi, \pi]$ [19]. As Ψ is increased, the rapidities move away from the real axis to form a curve C in the complex plane with endpoints $-a+\mathrm{i}b$ and $a+\mathrm{i}b$. In the thermodynamic limit the integral equations determining the ground-state energy are [3, 19, 20]

$$\phi(\gamma;\alpha) - \frac{1}{2\pi}\int_C \mathrm{d}\beta\,\phi(2\gamma;\alpha-\beta)R(\beta) = R(\alpha) \tag{10}$$

$$\lim_{N\to\infty}\frac{E_0}{N}=e_0=\cosh\gamma-\frac{2\sinh\gamma}{2\pi}\int_C \mathrm{d}\alpha\, R(\alpha)\phi(\gamma;\alpha). \tag{11}$$

Here the function $R(\alpha)$ is defined by

$$R(\alpha_l)=\lim_{N\to\infty}\frac{1}{N(\alpha_{l+1}-\alpha_l)}. \tag{12}$$

It describes the density of rapidities along the curve C and is to be determined by solving (10). The endpoints of C are functions of the field Ψ and the polarization $y=1-(2n/N)$ (eigenvalue of (S^z/N). They are implicitly determined by

$$p^0(a+\mathrm{i}b)-\frac{1}{2\pi}\int_C \mathrm{d}\beta\,\Theta(a+\mathrm{i}b-\beta)R(\beta)=-\mathrm{i}\Psi+\frac{\pi}{2}(1-y). \tag{13}$$

For Ψ less than some critical value Ψ_c, $a=\pi$ and $-\gamma<b<0$, so the analytical properties of (10) allow one to deform the contour C to a straight horizontal segment and the ground-state energy remains independent of Ψ [19],

$$e_0=\cosh\gamma-\sinh\gamma\sum_{n=-\infty}^{\infty}\frac{\exp(-\gamma|n|)}{\cosh\gamma n} \tag{14}$$

with the polarization remaining at $y=0$ and b given by [3]

$$\Psi=-b-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\sinh bn}{\cosh\gamma n}. \tag{15}$$

This agrees with the fact that the free energy of the asymmetric six-vertex model does not depend on Ψ in this region [3, 10, 11]. The critical value Ψ_c is reached at $b=-\gamma$. Note that the series still converges.

We now extend the old results to the calculation of the low-lying excitations. A complete study of the solutions of (6) is missing and would be desirable. Supported by numerical evidence we conjecture that, as for $\Psi=0$, the lowest excitations are given by holes in the ground-state distribution [11]. In particular we choose (independently of the parity of n) two sets of states, in the sector $n=(N/2)-1=n_0-1$, defined by quantum numbers $\{I_l\}$

$$I_l=I_l^0+\frac{\sigma}{2} \tag{16}$$

with $\sigma=\pm1$, but with one I_l^0 absent [11] (it is understood that in the sectors $n<(N/2)-1$, one might as well consider multi-hole excitations, whose energy would be the sum of single hole energies. This quasi-particle feature is typical of Bethe ansatz solvable models). The corresponding rapidities are shifted from their ground state set $\{\alpha_l^0\}$ to a new set $\{\alpha_l\}$ where one rapidity is missing (hole). To calculate the excitation energy and the difference in momentum between the ground state and the excited state, we define a 'shift' function [21]

$$j(\alpha_l)=\lim_{N\to\infty}\frac{\alpha_l-\alpha_l^0}{\alpha_{l+1}^0-\alpha_l^0} \tag{17}$$

and obtain the following equations in the thermodynamic limit

$$E_{\mathrm{exc}}-E_0=\Delta E=-2\sinh\gamma\int_C \mathrm{d}\alpha\,\phi'(\gamma;\alpha)j(\alpha)+2\sinh\gamma\,\phi(\gamma;\alpha^{(\mathrm{h})}) \tag{18}$$

$$\Delta P=\mathrm{i}\Psi+p^0(\alpha^{(\mathrm{h})})-\int_C \mathrm{d}\alpha\,\phi(\gamma;\alpha)j(\alpha) \tag{19}$$

$$j(\alpha)+\frac{1}{2\pi}\int_C \mathrm{d}\beta\,\phi(2\gamma;\alpha-\beta)j(\beta)=\frac{\sigma}{2}-\frac{1}{2\pi}\Theta(\alpha-\alpha^{(\mathrm{h})}) \tag{20}$$

where $\alpha^{(h)}$ is the position of the hole on the curve C and $j(\alpha)$ is to be determined by solving equation (20). It follows from (9) and (16) that $\Delta P \in [-\pi, 0]$ for $\sigma = 1$ and $\Delta P \in [0, \pi]$ for $\sigma = -1$.

When $-\gamma < b < 0$ the contour in (18)–(20) can still be deformed to a straight horizontal segment, after which these equations can be dealt with in the usual way [21]. When $b = -\gamma$, to avoid the pole of $\phi(\gamma, \alpha)$ at $\alpha = -\mathrm{i}\gamma$ one should close the contour by including the real segment $[-\pi, \pi]$ and the two vertical parts along $\mathrm{Re}(\alpha) = \pm\pi$. We note that this alternative detour could be adopted for $\Psi < \Psi_c$, too.

As in the case $\Psi = 0$, elliptic functions and elliptic integrals naturally appear in the final expressions. We define the elliptic modulus k by [11, 22]

$$\frac{\gamma}{\pi} = \frac{K'(k)}{K(k)} \tag{21}$$

where $K(k)$ and $K'(k)$ are the complete elliptic integrals of first kind with modulus k and k' with $k^2 + k'^2 = 1$ [11, 22]. Energy and momentum, as functions of $\alpha^{(h)}$, can be expressed in terms of the elliptic dn and am functions and it can be checked that $\mathrm{Re}(\Delta E)$ is non-negative. It is more instructive though to eliminate $\alpha^{(h)}$ from the two relations (18) and (19) and to look at the single particle dispersion relation which we find to be

$$\begin{aligned}\Delta E(\Delta P) = {} & \frac{2\sinh\gamma K(k)}{\pi}\left[(1 - k^2 \sin^2\left(\Delta P + \frac{\pi\sigma}{2} + \frac{\mathrm{i}\Psi}{2}\right)\right]^{\frac{1}{2}} \\ & + \frac{2\sinh\gamma K(k)}{\pi}\left[\left(1 - k^2\cosh^2\left(\frac{\Psi}{2}\right)\right)\right]^{\frac{1}{2}}.\end{aligned} \tag{22}$$

The mass gap is obtained by taking the minimum of this expression, i.e. by setting $\Delta P = 0$ (or $\pm\pi$) in (22). This is in accordance with our numerical solution of the Bethe ansatz equations which show that the lowest-lying excitations are obtained when the hole in the distribution of rapidities is lying at the endpoints of the curve C. We obtain, therefore,

$$m = \frac{4\sinh\gamma K(k)}{\pi}\left[\left(1 - k^2\cosh^2\left(\frac{\Psi}{2}\right)\right)\right]^{\frac{1}{2}} \tag{23}$$

which gives for the critical field

$$\cosh\left(\frac{\Psi_c}{2}\right) = k^{-1} \tag{24}$$

alternative to the expression obtained by taking $b = -\gamma$ in (15). The equality of (24) and (15) for $\Psi = \Psi_c$ has been checked numerically. They reproduce the same critical curve. Figure 2 shows this curve in the γ–Ψ plane. From (24), for large values of γ, we obtain a linear dependence $\Psi \sim -2\ln 2 + \gamma + \mathrm{O}(\mathrm{e}^{-\gamma})$, as can be seen from figure 2. The occurence of the transition is intuitively clear since the terms $\mathrm{e}^{\Psi}\sigma^+\sigma^-$ and $\mathrm{e}^{-\Psi}\sigma^-\sigma^+$ tend to destroy the antiferromagnetic order present at $\Psi = 0$.

As Ψ approaches the critical value from below we obtain

$$m \sim \frac{4\sinh\gamma K(k)}{\pi}(k')^{\frac{1}{2}}(\Psi_c - \Psi)^{\frac{1}{2}} + \mathrm{O}(\Psi_c - \Psi). \tag{25}$$

Alternatively, one can keep Ψ fixed and change γ, until the gap vanishes at $\gamma_c(\Psi)$. Since (21) defines a regular function $k(\gamma)$ such that $k \in (0, 1)$ as $\gamma \in (0, \infty)$, we have likewise

$$m \sim c_0(\gamma - \gamma_c(\Psi))^{\frac{1}{2}} + \mathrm{O}(\gamma - \gamma_c(\Psi)). \tag{26}$$

The identification $m \sim \xi^{-1}$ would, therefore, give a correlation length exponent $\nu = \frac{1}{2}$ because γ is a regular function of T [3]. We observe that the transition to a massless

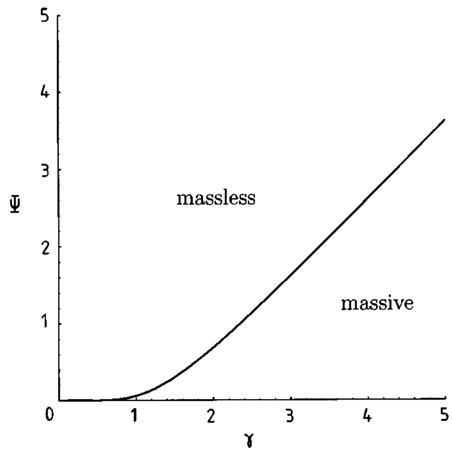

Figure 2. Phase diagram in the γ–Ψ plane; the full line corresponds to the critical line (24).

regime is different from the one occuring at $\Psi = 0, \gamma = 0$ (level 1 $SU(2)$ Wess–Zumino–Witten model [23]) where, as $\gamma \to 0^+$, the gap vanishes as

$$m = \frac{4 \sinh \gamma K(k)}{\pi} k' \sim 8\pi \exp(-\pi^2/2\gamma) \qquad \text{for } \gamma \to 0^+. \tag{27}$$

It is also instructive to compare (26) with the result at $V \neq 0$. From (3) it is clear that adding V brings a contribution $-2V$ to the excitations in the sector $n = n_0 - 1$. In this case the gap would vanish linearly in V at the critical value $V_c = m/2$, where m is given in (23). Therefore (26) describes a qualitatively new phase transition.

We now study the characteristics of the model *on* the critical line, i.e. at $\Psi = \Psi_c$. At small momenta we find for the dispersion relation

$$\Delta E \sim \frac{4 \sinh \gamma K(k)}{\pi} (k')^{\frac{1}{2}} (1 \pm \mathrm{i}) |\Delta P|^{\frac{1}{2}} \tag{28}$$

where positive (negative) imaginary part should be taken for $\Delta P > 0 (< 0)$. On a finite lattice, the momentum is quantized in units of $2\pi/N$, and (28) suggests that low-lying excitations should have gaps that scale like $N^{-\frac{1}{2}}$. This behaviour has been confirmed, at least for the real part, by solving (6) numerically up to 80 sites and diagonalizing (3), also numerically, up to 16 sites. We note that the $N^{-\frac{1}{2}}$ scaling differs from the one predicted for conformally invariant models where the energy gaps scale as N^{-1} [24]. Unusual exponents like $-\frac{3}{2}$ for the scaling of the gap with the lattice length have also been found in the ferromagnetic regime [14].

Above the curve (24) (see figure 2) we find a massless and conformal invariant phase. Our numerical solutions show indeed that this phase is characterized by a central charge $c = 1$. In the thermodynamic limit, the ground and excited states scale as N^{-1}, with a compactification radius R of a free massless boson field, which depends on Ψ and γ. It is interesting to note that approaching the curve $\Psi = \Psi_c(\gamma)$ the compactification radius takes the value $\frac{1}{2}$. The breaking of conformal invariance at this value of R can probably be related to a similar phenomenon occurring in the XXZ model in a magnetic field ($\Psi = 0$, $V \neq 0$) where conformal invariance is again broken, leading to a Prokovskii–Talapov phase transition curve [25]. This phenomenon takes place at $R = 1$ which is dual to our value $R = \frac{1}{2}$. The same phenomenon occurs for higher spin chains [26].

Our results lead to an interesting question, namely, how in the thermodynamic limit and in the context of the Coulomb gas picture can one account for the fact that a massless and conformal invariant phase ends on a curve which is also massless but exhibits different physics? A hint in this direction is the following observation. One expects this behaviour to be somehow manifest in terms of symmetry breaking. We have indeed noticed this numerically by diagonalizing (3) exactly for finite lattices. Above the curve (24) (see figure 2), in the massless and conformal invariant phase we find a high degeneracy in the energy spectrum which appears in the form of doublets and quadruplets. On the transition curve this degeneracy is lifted and many (but not all) of the doublets break up into singlets, while some quadruplets go over into a doublet plus singlets, a picture which remains valid in the whole massive phase. It is worth noting that this feature is already observable at finite lattice sizes.

This and other points are currently being investigated. The full details of our calculations will be given in a future publication.

After this work had been completed, Kim and Noh [27] analysed with more detail the operator content in the massless regime.

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